An amateur's view of concrete incompleteness and recent results of Harvey Friedman

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Gödel's first incompleteness theorem

Theorem (Gödel, 1931)

If T is a theory in a language L whose axioms are computably enumerable, and which interprets arithmetic, then there is an L-sentence G such that $T \nvdash G$ and $T \nvdash \neg G$.

Goodstein sequences

Form a sequence beginning with some natural number n (e.g $n = 19$). Begin by writing n in hereditary base 2 (e.g $G_1 = 19 = 2^{2^2} + 2 + 1$).

In general, write G_{m-1} in hereditary base m. Then, to form G_m, replace every occurrence of m with $m+1$, evaluate, and subtract 1.

Goodstein sequences grow quickly: $G_9 \approx 4.3 \times 10^{369693099}$.

Theorem 1 (Goodstein, 1944)

Every Goodstein sequence terminates at 0.

Theorem 2 (Kirby & Paris, 1982)

Theorem 1 is not provable in Peano Arithmetic.

Ramsey theory

Theorem 1 (Ramsey's theorem - finite version, 1930)

Suppose the subsets of $\mathbf{N}$ of size n are all coloured with one of k colours. For any N we want a monochromatic subset of $\{1, 2, \ldots, N\}$ of size at least s. Then there is a number $R(n, k, s)$ such that whenever $N \geq R(n, k, s)$ then $\{1, 2, \ldots, N\}$ has such a subset.

Theorem 2

Theorem 1 is provable in Peano Arithmetic.

The Paris-Harrington Principle

Definition

Call $A \subseteq \mathbf{N}$ *relatively big* if $|A| \geq \mathrm{Min}(A)$.

Theorem 1 (Paris-Harrington, 1977)

Suppose the subsets of $\mathbf{N}$ of size n are all coloured with one of k colours. For any N we want a **relatively big** monochromatic subset of $\{1, 2, \ldots, N\}$ of size at least s. Then there is a number $S(n, k, s)$ such that if $N \geq S(n, k, s)$ then $\{1, 2, \ldots, N\}$ has such a subset.

Theorem 2 (Paris-Harrington, 1977)

Theorem 1 is not provable in Peano Arithmetic.

Graph Embeddings

Theorem 1 (Kruskal, 1960)

Let $(T_i)_{i=1}^{\infty}$ be a sequence of finite trees. Then there are some $j < k$ such that $T_j \trianglelefteq T_k$. i.e there is an inf-preserving embedding $T_j \to T_k$.

Theorem 2 (Friedman, 2002)

For any n, there exists some N such that whenever $(T_i)_{i=1}^{N}$ is a sequence of finite trees where $|T_i| \leq n \cdot i$ for all i, then there are $j < k \leq N$, such that $T_j \trianglelefteq T_k$.

Theorem 3 (Friedman, 2002)

Neither theorem 1 nor 2 is provable in PA, or even ATR_0.

Arithmetical Hierarchy

A first-order formula is Σ_n^0 if it is of the form $\left(\exists \bar{x_1} \forall \bar{x_2} \ldots Q \bar{x_n}\right)\phi$, where ϕ contains only bounded quantifiers.

A first-order formula is Π_n^0 if it is of the form $\left(\forall \bar{x_1} \exists \bar{x_2} \ldots Q \bar{x_n}\right)\phi$, where ϕ contains only bounded quantifiers.

$\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$

This produces a hierarchy:

$$\Delta_0^0 \subset \Delta_1^0 \begin{array}{c} \subset \\ \subset \end{array} \begin{array}{c} \Sigma_1^0 \\ \Pi_1^0 \end{array} \begin{array}{c} \subset \\ \subset \end{array} \Delta_2^0 \begin{array}{c} \subset \\ \subset \end{array} \begin{array}{c} \Sigma_2^0 \\ \Pi_2^0 \end{array} \begin{array}{c} \subset \\ \subset \end{array} \Delta_3^0 \ldots$$

Facts

- The Δ^0_0-sets are exactly the *primitive recursive* sets.
- The Δ^0_1-sets are exactly the *computable* sets.
- The Σ^0_1-sets are exactly the *computably enumerable* sets.
- Post's theorem: the Σ^0_{n+1}-sets are exactly the $\emptyset^{(n)}$-enumerable sets
- Peano Arithmetic is Σ^0_1-complete.
- Fermat's Last Theorem and the Riemann Hypothesis are Π^0_1
- Finite Kruskal's theorem, the Paris-Harrington principle, and the termination of Goodstein sequences are all Π^0_2

Proof theoretic hierarchy

PRA (Primitive recursive arithmetic)
ACA_0 (Arithmetical comprehension, incorporating PA) $\leftarrow$ Most finite mathematics
ACA' (ACA_0 + "$\forall n \forall x \subseteq \omega$, the nth Turing jump of x exists") $\leftarrow$ Paris-Harrington & Goodstein sequences
ATR_0 (Arithmetical transfinite recursion) $\leftarrow$ Much analysis and topology
Π^1_1-CA_0 (Π^1_1-comprehension) $\Leftarrow$ Kruskal-type theorems
Z_2 (Second order arithmetic, or Π^1_∞-CA_0)
Z_3
ZFC $\leftarrow$ Almost all mathematics
ZFC + $\exists$ an inaccessible cardinal
ZFC + $\exists$ a strongly 1-Mahlo cardinal
ZFC + $\forall n$ $\exists$ a strongly n-Mahlo cardinal $\leftarrow$ BRT
ZFC + $\exists$ a huge cardinal

A little set theory

Definition

- A cardinal κ is (strongly) *0-Mahlo* if it is (strongly) inaccessible.
- κ is (strongly) *n+1-Mahlo* if $\{\lambda : \lambda < \kappa \text{ is (strongly) } n\text{-Mahlo}\}$ is stationary in κ.

Complementation theorem

Let SD be the class of functions $f : \mathbf{N^k} \to \mathbf{N}$ which are *strictly dominating*, that is $f(\bar{x}) > \mathrm{Max}(\bar{x})$.

Theorem (Complementation theorem)

For any $f \in SD$ there exists an infinite set $A \subseteq \mathbf{N}$ such that $f(A^k) = \mathbf{N} \backslash A$.

Boolean relation theory

Let ELG be the class of functions $f : \mathbf{N^k} \to \mathbf{N}$ which exhibit *expansive linear growth*, that is there are $C, D > 1$ so that for all but finitely many $\bar{x}$,

$$C \cdot \mathrm{Max}(\bar{x}) \leq f(\bar{x}) \leq D \cdot \mathrm{Max}(\bar{x})$$

Theorem (Friedman, 2009)

Given $f, g \in ELG$ there are infinite $A, B, C \subseteq \mathbf{N}$, so

$$A \sqcup f(A) \subseteq C \sqcup g(B)$$
$$\textit{and} \quad A \sqcup f(B) \subseteq C \sqcup g(C)$$

This requires ZFC $+ \ \forall k \ \ \exists$ a strongly k-Mahlo cardinal. Replacing the occurrences of A, B, C above produces $3^8 = 6561$ different statements. All but 12 are provable or refutable within PA. These 12 exotic cases require Mahlo cardinals.

The exotic cases of Boolean relation theory "are Π^0_2 ".

These cases involve functions on **N**, which exhibit *expansive linear growth*, which pick out configurations of *infinite* sets. Other possible settings include:

- Replacing **N** with **Z**, **Q**, **R**, **C**, or more general spaces
- Functions satisfying various analytic or topological conditions
- Replacing infinite sets with those satisfying various geometric or topological conditions

For example, consider the class V of all bounded linear operators on a Hilbert space H, and privileged sets comprising the class K of non-trivial closed subspaces of H. Then the statement

$$\forall f \in V \;\; \exists A \in K \;\; f(A) \subseteq A$$

is the *invariant subspace problem* for H.

Kernel Structures

Goal (Friedman)

"To find an explicitly Π^0_1 sentence which can only be proved using large cardinals, and which arguably represents clear and compelling information in the finite mathematical realm."

We consider directed graphs $(\mathbf{Q}^k, R)$ where $R \subseteq \mathbf{Q}^{2k}$. Such a graph is...

- *order invariant* if whenever $\bar{a}, \bar{b} \in \mathbf{Q}^{2k}$ are order-equivalent, then $\bar{a} \in R \Leftrightarrow \bar{b} \in R$
- *downward* if $\bar{x} R \bar{y} \implies \mathrm{Max}(\bar{x}) > \mathrm{Max}(\bar{y})$

We work with some fixed $(\mathbf{Q}^k, R)$. Given $A \subseteq \mathbf{Q}^k$, then $B \subseteq A$ is a *kernel* of A if

- $\forall a \in A \backslash B \quad \exists b \in B \quad aRb$
- $\nexists b_1, b_2 \in B \quad b_1 R b_2$

Given $C \subseteq \mathbf{Q}^{3k}$, then $A \subseteq \mathbf{Q}^{\mathbf{k}}$ is *C-closed* if $C[A^2] \subseteq A$.

Now define two specific subsets of $\mathbf{Q}^{3k}$:

- $P(x, y, z) \Leftrightarrow$ every coordinate of z is a coordinate of x or y

Given $x \in \mathbf{Q}^k$, the *upper shift* of x is found by adding 1 to all its non-negative coordinates

- $J(x, y, z) \Leftrightarrow z$ is the upper shift of either x or y

Goal achieved!

The independent statement (Friedman, 2010)

For every downward, order invariant digraph on $\mathbf{Q}^k$, there is a non-empty P-closed subset which has a J-closed kernel.

The above statement "is Π^0_1" and can only be proved in ZFC + "$\forall k \quad \exists$ an infinite cardinal with the k-Stationary Ramsey Property", but not in any weaker system.

Definition

A cardinal κ has the k-SRP if for every partition of the collection of its k-subsets into two parts A and B, either A or B contains a stationary subset.